Pontryagin’s maximum principle for optimal control of SPDEs

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Outline

1. Problem formulation
   - Statement
   - Assumptions
   - The adjoint equation

2. Pontryagin stochastic maximum principle

3. Plan of the proof
   - Estimates
   - Variational inequality
     - Proof
   - Two more lemmas
   - Proof of main result
Consider the SPDE on a sep. Hilbert space $K$:

\[
\begin{aligned}
\left\{ \begin{array}{l}
dX^{u(\cdot)}(t) &= \left(A(t)X^{u(\cdot)}(t) + F(X^{u(\cdot)}(t), u(t))\right)dt + G(X^{u(\cdot)}(t))dM(t), \\
X^{u(\cdot)}(0) &= x.
\end{array} \right.
\end{aligned}
\] (1)

- $M$ is a continuous martingale, $<< M >>_t = \int_0^t Q(s) \, ds$, for a predictable process $Q(\cdot)$ s.t. $Q(t) \in L_1(K)$ is symmetric, positive definite, $Q(t) \leq Q$, where $Q \in L_1(K)$ (positive definite).

- $F : K \times \mathcal{O} \to K$ ($\mathcal{O}$ is a sep. Hilbert space).

- $G : K \to L_2(K_0, K)$, $K_0 = Q^{-1/2}(K)$.

- $A(t, \omega)$ is a predictable unbounded linear operator on $K$. 
We shall derive a stochastic maximum principle for this control problem by using the adjoint equation (BSPDE).

- \( u(\cdot) : [0, T] \times \Omega \rightarrow \mathcal{O} \) is admissible if \( u(\cdot) \in L^2_{\mathcal{F}}(0, T; \mathcal{O}) \) and \( u(t) \in U \) a.e., a.s. (\( U \) is a nonempty convex subset of \( \mathcal{O} \)).

The set of admissible controls \( \sim \mathcal{U}_{ad} \).

\[ L^2_{\mathcal{F}}(0, T; E) := \{ \psi : [0, T] \times \Omega \rightarrow E, \text{predictable,} \]
\[ \mathbb{E}[\int_0^T |\psi(t)|_E^2 dt] < \infty \}. \]
We shall derive a **stochastic maximum principle** for this control problem by using the adjoint equation (BSPDE).

- **u(·) : [0, T] × \Omega → \mathcal{O}** is admissible if \( u(\cdot) \in L^2_\mathcal{F}(0, T; \mathcal{O}) \) and \( u(t) \in U \ a.e., a.s. \) (\( U \) is a nonempty convex subset of \( \mathcal{O} \)).

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Define

- the cost functional:

\[ J(u(\cdot)) := \mathbb{E} \left[ \int_0^T g(X^{u(\cdot)}(t), u(t)) \, dt + \phi(X^{u(\cdot)}(T)) \right], \]

\[ J^* := \inf \{ J(u(\cdot)) : u(\cdot) \in \mathcal{U}_{ad} \}. \]

The control problem for this SPDE (1) is to find a control \( u^*(\cdot) \) and the corresponding solution \( X^{u^*(\cdot)} \) of (1) s.t.

\[ J^* = J(u^*(\cdot)). \]

- \( u^*(\cdot) \) is an optimal control.
- \( X^{u^*(\cdot)} \) (or briefly \( X^* \)) is an optimal solution.
- The pair \( (X^*, u^*(\cdot)) \) is an optimal pair.
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Problem formulation

Pontryagin stochastic maximum principle

Plan of the proof

Statement
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-The adjoint equation

(H1) $F, G, g, \phi$ are $C^1$ w.r.t. $x$, $F$ is $C^1$ w.r.t. $u$, the derivatives $F_x, F_u, G_x, g_x$ are uniformly bounded.

Also $|\phi_x|_K \leq C_1 (1 + |x|_K)$, some constant $C_1 > 0$.

(H2) $g_x$ satisfies Lipschitz condition with respect to $u$ uniformly in $x$.

(H3) $A(t, \omega)$ is a predictable linear operator on $K$, belongs to $L(V; V')$ ($(V, K, V')$ is a Gelfand triple),

1. $2 \langle A(t, \omega) y, y \rangle + \alpha |y|_V^2 \leq \lambda |y|^2 \quad \text{a.e. } t \in [0, T], \text{ a.s. } \forall y \in V,$ for some $\alpha, \lambda > 0$.

2. $\exists C_2 \geq 0 \text{ s.t. } |A(t, \omega) y|_{V'} \leq C_2 |y|_V \quad \forall (t, \omega), \forall y \in V.$
Define the **Hamiltonian**:

$$ H : [0, T] \times \Omega \times K \times \mathcal{O} \times K \times L_2(K) \to \mathbb{R}, $$

$$ H(t, x, u, y, z) := -g(x, u) + \langle F(x, u), y \rangle + \langle G(x)Q^{1/2}(t), z \rangle. $$

The (adjoint) BSPDE:

$$ -dY^{u(\cdot)}(t) = \left[ A^*(t) Y^{u(\cdot)}(t) + \nabla_x H(X^{u(\cdot)}(t), u(t), Y^{u(\cdot)}(t), Z^{u(\cdot)}(t)Q^{1/2}(t)) \right] dt $$

$$ -Z^{u(\cdot)}(t) dM(t) - dN^{u(\cdot)}(t), \quad 0 \leq t \leq T, $$

$$ Y^{u(\cdot)}(T) = -\nabla \phi(X^{u(\cdot)}(T)), $$

$A^*(t)$ is the adjoint operator of $A(t)$. 
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Y^{u(\cdot)}(T) = -\nabla \phi(X^{u(\cdot)}(T)),
\]

\(A^*(t)\) is the adjoint operator of \(A(t)\).
\[ \mathcal{M}^{2,c}_{[0,T]}(K) \sim \text{the space of continuous square integrable martingales in } K. \]

Two elements \( M \) and \( N \) of \( \mathcal{M}^{2,c}_{[0,T]}(K) \) are very strongly orthogonal (VSO) if

\[
\mathbb{E} [M(\tau) \otimes N(\tau)] = \mathbb{E} [M(0) \otimes N(0)],
\]

for all \([0, T]\)-valued stopping times \( \tau \).

In fact: \( M \) and \( N \) are VSO \( \iff \ll M, N >> = 0. \)

\[ \Lambda^2(K; \mathcal{P}, M) \sim \text{the space of integrands w.r.t. } M \text{ s.t.} \]

\[
\Phi(t, \omega) \, Q^{1/2}(t, \omega) \in L_2(K), \text{ for every } h \in K
\]

the \( K \)-valued process \( \Phi \circ Q^{1/2}(h) \) is predictable,

\[
\mathbb{E} \left[ \int_0^T \| (\Phi \circ Q^{1/2})(t) \|_2^2 \, dt \right] < \infty.
\]
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Definition

A solution of a BSPDE:

\[
\begin{aligned}
- dY(t) &= \left( A(t) Y(t) + f(t, Y(t), Z(t) Q^{1/2}(t)) \right) dt \\
- Z(t) dM(t) - dN(t), & \quad 0 \leq t \leq T, \\
Y(T) &= \xi,
\end{aligned}
\]

is \((Y, Z, N) \in L^2_{\mathcal{F}}(0, T; V) \times \Lambda^2(K; \mathcal{P}, \mathcal{M}) \times \mathcal{M}^{2,c}_{[0, T]}(K)\) s.t. \(\forall t \in [0, T] :\)

\[
Y(t) = \xi + \int_t^T (A(s) Y(s) + f(s, Y(s), Z(s) Q^{1/2}(s))) ds \\
- \int_t^T Z(s) dM(s) - \int_t^T dN(s),
\]

\(N(0) = 0, \) \(N\) is VSO to \(M.\)
The following theorem gives the unique solution to the adjoint equation.

**Theorem 1 (Existence & uniqueness of the solution of the adjoint BSPDE)**

Assume (H1)–(H3). There exists a unique solution \((Y^{u(\cdot)}, Z^{u(\cdot)}, N^{u(\cdot)})\) to the adjoint BSPDE in \(L^2_{\mathcal{F}}(0, T; V) \times \Lambda^2(K; \mathcal{P}, \mathcal{M}) \times \mathcal{M}^{2,c}_{[0,T]}(K)\).

The proof of this theorem can be found in:


Denote the solution of the (adjoint) BSPDE corresponding to \(u^*(\cdot)\) by \((Y^*, Z^*, N^*)\).
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The proof of this theorem can be found in: [ Al-Hussein, A., Backward stochastic partial differential equations driven by infinite dimensional martingales and applications, Stochastics, Vol. 81, No. 6, 2009, 601-626 ].

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Theorem 2 (Pontryagin stochastic maximum principle)

Suppose (H1)–(H3). Assume \((X^*, u^*(\cdot))\) is an optimal pair for our control problem.

Then there exists a unique solution \((Y^*, Z^*, N^*)\) to the corresponding BSPDE s.t. the following inequality holds:

\[
\left\langle \nabla_u H(t, X^*(t), u^*(t), Y^*(t), Z^*(t)Q^{1/2}(t)), u - u^*(t) \right\rangle_\mathcal{F} \leq 0,
\]

\(\forall \ u \in U, \ \text{a.e.} \ t \in [0, T], \ \text{a.s.}\)
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Let $u^*(\cdot)$ be an optimal control and $X^*$ be the corresponding solution of SPDE (1). Let $u(\cdot)$ be s.t. $u^*(\cdot) + u(\cdot) \in \mathcal{U}_{ad}$.

For $0 \leq \varepsilon \leq 1$ consider the variational control:

$$u_\varepsilon(t) = u^*(t) + \varepsilon u(t), \quad t \in [0, T].$$

The convexity of $U \Rightarrow u_\varepsilon(\cdot) \in \mathcal{U}_{ad}$.

Get the corresponding $X_\varepsilon$ of (1).

Let $p$ be the solution of the linear equation:

$$\begin{cases} 
    dp(t) = (A(t)p(t) + F_x(X^*(t), u^*(t))p(t))dt \\
    \quad + F_u(X^*(t), u^*(t))u(t)dt + G_x(X^*(t))p(t)dM(t), \\
    p(0) = 0.
\end{cases}$$
Let \( u^*(\cdot) \) be an optimal control and \( X^* \) be the corresponding solution of SPDE (1). Let \( u(\cdot) \) be s.t. \( u^*(\cdot) + u(\cdot) \in U_{ad} \).

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    p(0) = 0.
\end{cases}
\]
We obtain:

**Lemma 1**

Assume (H1), (H3). Let

\[ \eta_\varepsilon(t) = \frac{X_\varepsilon(t) - X^*(t)}{\varepsilon} - p(t), \quad t \in [0, T]. \]

Then:

1. \[ \sup_{t \in [0, T]} E \left[ |p(t)|^2 \right] < \infty, \]
2. \[ \sup_{t \in [0, T]} E \left[ |X_\varepsilon(t) - X^*(t)|^2 \right] = O(\varepsilon^2), \]
3. \[ \lim_{\varepsilon \to 0^+} \sup_{t \in [0, T]} E \left[ |\eta_\varepsilon(t)|^2 \right] = 0. \]
Lemma 2 (Variational inequality)

Suppose (H1)–(H3). Then \( \forall \varepsilon > 0 \),

\[
J(u_\varepsilon(\cdot)) - J(u^*(\cdot)) = \varepsilon \mathbb{E} \left[ \phi_X(X^*(T)) p(T) \right]
\]

\[
+ \varepsilon \mathbb{E} \left[ \int_0^T g_X(X^*(s), u^*(s)) p(s) \, ds \right]
\]

\[
+ \mathbb{E} \left[ \int_0^T (g(X^*(s), u_\varepsilon(s)) - g(X^*(s), u^*(s))) \, ds \right]
\]

\[
+ o(\varepsilon).
\]
Sketch proof of Lemma 2

Note

\[ J(u_\varepsilon(\cdot)) - J(u^*(\cdot)) = I_1(\varepsilon) + I_2(\varepsilon), \]

where

\[ I_1(\varepsilon) = \mathbb{E} \left[ \phi(X_\varepsilon(T)) - \phi(X^*(T)) \right], \]

\[ I_2(\varepsilon) = \mathbb{E} \left[ \int_0^T \left( g(X_\varepsilon(s), u_\varepsilon(s)) - g(X^*(s), u^*(s)) \right) ds \right]. \]

Then apply Lemma 1 (2), (3).
Lemma 3

If (H1)–(H3) hold, then

\[- \varepsilon \mathbb{E} \langle Y^*(T), p(T) \rangle + \varepsilon \mathbb{E} \left[ \int_0^T g_x(X^*(s), u^*(s)) p(s) \, ds \right] + \mathbb{E} \left[ \int_0^T ( -\delta_\varepsilon H(s) + \langle Y^*(s), \delta_\varepsilon F(s) \rangle ) \, ds \right] \geq o(\varepsilon),\]

where

\[
\delta_\varepsilon F(s) = F(X^*(s), u_\varepsilon(s)) - F(X^*(s), u^*(s)),
\]

\[
\delta_\varepsilon H(s) = H(X^*(s), u_\varepsilon(s), Y^*(s), Z^*(s)Q^{1/2}(s)) - H(X^*(s), u^*(s), Y^*(s), Z^*(s)Q^{1/2}(s)).
\]
Proof of Lemma 3

Since $u^*(\cdot)$ is an optimal control, $J(u_\varepsilon(\cdot)) - J(u^*(\cdot)) \geq 0$.

Next apply Lemma 2 and the definition of the Hamiltonian $H$.

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Lemma 4

\[ \mathbb{E} \langle Y^*(T), p(T) \rangle = \mathbb{E} \left[ \int_0^T g_x(X^*(s), u^*(s)) p(s) \, ds \right] \]
\[ + \mathbb{E} \left[ \int_0^T \langle Y^*(s), F_u(X^*(s), u^*(s))u(s) \rangle \, ds \right]. \]

Proof of Lemma 4

▷ Use Itô's formula together with

\[ \langle \nabla_x H(X^*(t), u^*(t), Y^*(t), Z^*(t) Q^{1/2}(t)), p(t) \rangle \]
\[ = - g_x(X^*(t), u^*(t)) p(t) + \langle F_x(X^*(t), u^*(t)) p(t), Y^*(t) \rangle \]
\[ + \langle G(X^*(t)) Q^{1/2}(t), Z^*(t) Q^{1/2}(t) \rangle \quad a.s. \forall t \in [0, T]. \]
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= - g_x(X^*(t), u^*(t)) p(t) + \left\langle F_x(X^*(t), u^*(t)) p(t), Y^*(t) \right\rangle \\
+ \left\langle G(X^*(t))Q^{1/2}(t), Z^*(t)Q^{1/2}(t) \right\rangle_2 \quad \text{a.s. } \forall \ t \in [0, T].
\]
Proof of main result

Consider the adjoint equation. From Lemma 3, Lemma 4 get

\[
\mathbb{E} \left[ \int_0^T \langle Y^*(s), \delta \epsilon F(s) - \epsilon F_u(X^*(s), u^*(s))u(s) \rangle ds \right] \\
- \mathbb{E} \left[ \int_0^T \delta \epsilon H(s) ds \right] \geq o(\epsilon).
\]

The continuity, boundedness of \( F_u \) in (H1) and the DCT give

\[
\frac{1}{\epsilon} \mathbb{E} \left[ \int_0^T \langle Y^*(s), \delta \epsilon F(s) - \epsilon F_u(X^*(s), u^*(s))u(s) \rangle ds \right] \\
= \mathbb{E} \left[ \int_0^T \langle Y^*(s), \int_0^1 \left( F_u(X^*(s), u^*(s) + \theta(u_\epsilon(s) - u^*(s))) \\
- F_u(X^*(s), u^*(s)) \right) u(s) d\theta \rangle ds \right] \to 0,
\]
as \( \epsilon \to 0^+ \).
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- F_u(X^*(s), u^*(s)) \right) u(s) d\theta \rangle ds \right] \to 0,
\]
as \( \varepsilon \to 0^+ \).
In particular

\[ \mathbb{E} \left[ \int_0^T \left\langle Y^*(s), \delta_\varepsilon F(s) - \varepsilon F_u(X^*(s), u^*(s)) u(s) \right\rangle ds \right] = o(\varepsilon). \]

\[ \therefore - \mathbb{E} \left[ \int_0^T \delta_\varepsilon H(s) ds \right] \geq o(\varepsilon). \]

\[ \therefore \text{by dividing this inequality by } \varepsilon \text{ and letting } \varepsilon \to 0^+: \]

\[ \mathbb{E} \left[ \int_0^T \nabla_u H(t, X^*(t), u^*(t), Y^*(t), Z^*(t) Q^{1/2}(t)), u(t) \right] dt \leq 0. \]

The theorem then follows.
Thank you